

# Polyhedral Realizations of Crystal Bases and Braid-type Isomorphisms

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## 1 Introduction

In [5],[6], Kashiwara introduced the theory of crystal base. He has shown that the existence of crystal base for the subalgebra  $U_q^-(\mathfrak{g})$  of the given quantum algebra  $U_q(\mathfrak{g})$  and arbitrary integrable highest weight  $U_q(\mathfrak{g})$ -modules.

It is a fundamental problem to present a concrete realization of crystal bases for given integrable highest weight modules as explicitly as possible. Up to the present time, there are several kinds of realizations, e.g., for finite types, some analogues of Young tableaux were introduced ([11]) and the piece-wise linear combinatorics for type  $A$  were introduced in [2]; for affine  $A$  type, the realization of crystal bases using Young diagram were treated in [3] and for general affine cases the new tools 'perfect crystals' and 'path realization' were invented ([9][10]), and for general Kac-Moody cases in [12][13] Littelmann realized the crystal base for symmetrizable Kac-Moody Lie algebras and in [16][17] the 'polyhedral realization' was introduced. A particular feature of the polyhedral realization is that it has a very explicit form and it is not necessary to distinguish the type of the underlying Kac-Moody algebra. But this realization makes sense under the assumption called 'ample' (see 3.2). We found some examples which does not hold the assumption. In order to avoid this difficulties, at least in semi-simple cases, we introduce the 'braid-type isomorphism' (see also [14]).

In the former half of this manuscript, we shall review the polyhedral realizations of crystal bases ([16],[17]). It can be simply understood as the following thing; the crystal base of given module  $V(\lambda)$  or subalgebra  $U_q^-(\mathfrak{g})$  is realized as the set of lattice points in some convex polytope or polyhedral convex cone in the infinite dimensional vector space  $\mathbf{Q}^\infty$ . To be more precise, we prepare some ingredients here. Let  $B_i$  ( $i \in I$ ) be the crystal associated with the simple root  $\alpha_i$ , where  $I$  is the finite index set of simple roots (see 2.2) and  $B(\lambda)$  be the crystal base of the irreducible highest weight module  $V(\lambda)$  ( $\lambda \in P_+$ ). Then we have the embedding of crystal (see 3.2):

$$\Psi^{(\lambda)} : B(\lambda) \hookrightarrow \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1} \otimes R_\lambda (\cong \mathbf{Z}^\infty), \quad (*)$$

where  $\iota = \cdots i_k \cdots i_2 i_1$  is an infinite sequence of indices in  $I$ . Here note that each  $B_{i_k}$  is identified with the set of integers  $\mathbf{Z}$  as a set and the crystal  $R_\lambda$  includes only single element. Thus, we can identify  $\mathbf{Z}^\infty$  with the RHS of (\*). Under the assumption 'ample', the image of  $B(\lambda)$  in  $\mathbf{Z}^\infty$  is obtained as the set of lattice

points in the convex polyhedron defined by some system of linear inequalities (see Sect.3). This is the reason why it is called “polyhedral”.

As we have mentioned above, in the construction of this realization, we do not need to distinguish the types of the associated Lie algebras, like ‘finite’, ‘affine’, ‘hyperbolic’.... We only need the Cartan data as the Kac-Moody algebra. Indeed, we shall see that our realization can be applied to arbitrary rank 2 Kac-Moody algebras.

In the latter half of the manuscript, we shall show the existence of the braid-type isomorphisms between the tensor products of crystals  $B_i$ ’s and  $B_j$ ’s in the cases  $\langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle = 0, 1, 2, 3$ . The explicit form of this isomorphisms is not simple, indeed, they are expressed by several piece-wise linear functions. We will know, however, that it is natural in the theory of crystals. As their application, we shall show that if  $\mathfrak{g}$  is semi-simple, the crystal  $B(\lambda)$  can be realized in the finite rank  $\mathbf{Z}$ -lattice and the rank is equal to the length of the longest element in the corresponding Weyl group (see Proposition 4.1). By the braid-type isomorphism, we can manage to treat some ‘non-ample’ cases (see 4.3).

## 2 Crystal Bases and Crystals

### 2.1 Definition of crystal bases

We review the crystal bases for integrable highest weight modules and the nilpotent subalgebra  $U_q^-(\mathfrak{g})$  which are our main subject of study. All the results in this subsection related to crystal bases are due to M.Kashiwara [6].

Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra over  $\mathbf{Q}$  with a Cartan subalgebra  $\mathfrak{t}$ , a weight lattice  $P \subset \mathfrak{t}^*$ , the set of simple roots  $\{\alpha_i : i \in I\} \subset \mathfrak{t}^*$ , and the set of coroots  $\{h_i : i \in I\} \subset \mathfrak{t}$ , where  $I$  is a finite index set. Let  $\langle h, \lambda \rangle$  be the pairing between  $\mathfrak{t}$  and  $\mathfrak{t}^*$ , and  $(\alpha, \beta)$  be an inner product on  $\mathfrak{t}^*$  such that  $(\alpha_i, \alpha_i) \in 2\mathbf{Z}_{\geq 0}$  and  $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$  for  $\lambda \in \mathfrak{t}^*$ . Let  $P^* = \{h \in \mathfrak{t} : \langle h, P \rangle \subset \mathbf{Z}\}$  and  $P_+ := \{\lambda \in P : \langle h_i, \lambda \rangle \in \mathbf{Z}_{\geq 0}\}$ . We call an element in  $P_+$  a *dominant integral weight*. The quantum algebra  $U_q(\mathfrak{g})$  is an associative  $\mathbf{Q}(q)$ -algebra generated by the  $e_i, f_i$  ( $i \in I$ ), and  $q^h$  ( $h \in P^*$ ) satisfying the usual relations (see e.g., [6] or [16]). The algebra  $U_q^-(\mathfrak{g})$  is the subalgebra of  $U_q(\mathfrak{g})$  generated by the  $f_i$  ( $i \in I$ ).

Let  $V(\lambda)$  be the irreducible highest weight module of  $U_q(\mathfrak{g})$  with the highest weight  $\lambda \in P_+$ . It can be defined by

$$V(\lambda) := U_q(\mathfrak{g}) / \left( \sum_i U_q(\mathfrak{g})e_i + \sum_i U_q(\mathfrak{g})f_i^{\langle h_i, \lambda \rangle + 1} + \sum_{h \in P^*} U_q(\mathfrak{g})(q^h - q^{\langle h, \lambda \rangle}) \right).$$

It is well-known that as a  $U_q^-(\mathfrak{g})$ -module, there is the following natural isomorphism:

$$V(\lambda) \cong U_q^-(\mathfrak{g}) / \sum_i U_q^-(\mathfrak{g})f_i^{\langle h_i, \lambda \rangle + 1}.$$

Let  $\pi_\lambda$  be a natural projection  $U_q^-(\mathfrak{g}) \longrightarrow V(\lambda)$  and set  $u_\lambda := \pi_\lambda(1)$ . This is the unique highest weight vector in  $V(\lambda)$  up to constant. We also denote the unit  $1 \in U_q^-(\mathfrak{g})$  by  $u_\infty$ , namely,  $u_\lambda = \pi_\lambda(u_\infty)$ .

Let  $\mathcal{O}_{\text{int}}(\mathfrak{g})$  be the category of upper-bounded integrable modules (see [6]). This category is semi-simple and each simple object is isomorphic to some  $V(\lambda)$ . For an object in  $\mathcal{O}_{\text{int}}(\mathfrak{g})$  (resp.  $U_q^-(\mathfrak{g})$ ) and any  $i \in I$ , we have the decomposition:  $V = \bigoplus_n f_i^{(n)}(\text{Ker } e_i)$  (resp.  $U_q^-(\mathfrak{g}) = \bigoplus_n f_i^{(n)}(\text{Ker } e'_i)$ ) (as for  $e'_i$  see [6],[16]). Using this, we can define the endomorphisms  $\tilde{e}_i$  and  $\tilde{f}_i \in \text{End}(V)$  (resp.  $\text{End}(U_q^-(\mathfrak{g}))$ ) by

$$\tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u, \text{ and } \tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \text{ for } u \in \text{Ker } e_i \text{ (resp. Ker } e'_i),$$

where we understand that  $\tilde{e}_i u = 0$  for  $u \in \text{Ker } e_i$  (resp.  $\text{Ker } e'_i$ ). Let  $A \subset \mathbf{Q}(q)$  be the subring of rational functions regular at  $q = 0$ .

In the following definition, let  $M$  be an object in  $\mathcal{O}_{\text{int}}(\mathfrak{g})$  or  $U_q^-(\mathfrak{g})$ .

**Definition 2.1 ([6])** *A pair  $(L, B)$  is a crystal base of  $M$ , if it satisfies:*

- (i)  $L$  is a free  $A$ -submodule of  $M$  and  $M \cong \mathbf{Q}(q) \otimes_A L$ .
- (ii)  $L = \bigoplus_{\lambda \in P} L_\lambda$  and  $B = \sqcup_{\lambda \in P} B_\lambda$  where  $L_\lambda := L \cap M_\lambda$  and  $B_\lambda := B \cap L_\lambda / qL_\lambda$ .
- (iii)  $B$  is a basis of the  $\mathbf{Q}$ -vector space  $L/qL$ .
- (iv)  $\tilde{e}_i L \subset L$  and  $\tilde{f}_i L \subset L$ .  
By (iv) the  $\tilde{e}_i$  and the  $\tilde{f}_i$  act on  $L/qL$  and
- (v)  $\tilde{e}_i B \subset B \sqcup \{0\}$  and  $\tilde{f}_i B \subset B \sqcup \{0\}$ .
- (vi) For  $u, v \in B$ ,  $\tilde{f}_i u = v$  if and only if  $\tilde{e}_i v = u$ .

We set

$$L(\lambda) := \sum_{i_j \in I, l \geq 0} A \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\lambda, \quad (2.1)$$

$$B(\lambda) := \{ \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\lambda \bmod qL(\lambda) \mid i_j \in I, l \geq 0 \} \setminus \{0\}. \quad (2.2)$$

Here the definition of  $L(\infty)$  and  $B(\infty)$  are given by replacing  $u_\lambda$  by  $u_\infty$  in (2.1) and (2.2).

**Theorem 2.2 ([6])** *The pair  $(L(\lambda), B(\lambda))$  (resp.  $(L(\infty), B(\infty))$ ) is the crystal base of  $V(\lambda)$  (resp.  $U_q^-(\mathfrak{g})$ ).*

## 2.2 Definition of crystals

A crystal is a combinatorial object obtained by abstracting the properties of crystal bases. In what follows we fix a finite index set  $I$  and a weight lattice  $P$  as above.

**Definition 2.3** *A crystal  $B$  is a set endowed with the following maps:*

$$wt : B \longrightarrow P, \quad (2.3)$$

$$\varepsilon_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\} \text{ for } i \in I, \quad (2.4)$$

$$\tilde{e}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \text{ for } i \in I. \quad (2.5)$$

Here 0 is an ideal element which is not included in  $B$ . Indeed,  $B$  is originally a basis of a linear space, which does not include the zero vector. This 0 plays the similar role to the zero vector. These maps must satisfy the following axioms: for all  $b, b_1, b_2 \in B$ , we have

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle, \quad (2.6)$$

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \quad (2.7)$$

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \quad (2.8)$$

$$\tilde{e}_i b_2 = b_1 \text{ if and only if } \tilde{f}_i b_1 = b_2, \quad (2.9)$$

$$\text{if } \varepsilon_i(b) = -\infty, \text{ then } \tilde{e}_i b = \tilde{f}_i b = 0, \quad (2.10)$$

$$\tilde{e}_i(0) = \tilde{f}_i(0) = 0. \quad (2.11)$$

The above axioms allow us to make a crystal  $B$  into a colored oriented graph with the set of colors  $I$ .

**Definition 2.4** The crystal graph of a crystal  $B$  is a colored oriented graph given by the rule :  $b_1 \xrightarrow{i} b_2$  if and only if  $b_2 = \tilde{f}_i b_1$  ( $b_1, b_2 \in B$ ).

**Definition 2.5** (i) Let  $B_1$  and  $B_2$  be crystals. A strict morphism of crystals  $\psi : B_1 \longrightarrow B_2$  is a map  $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$  satisfying:  $\psi(0) = 0$ , if  $b \in B_1$  and  $\psi(b) \in B_2$ , then

$$wt(\psi(b)) = wt(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \quad (2.12)$$

and the map  $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$  commutes with all  $\tilde{e}_i$  and  $\tilde{f}_i$ .

(ii) An injective (resp. bijective) strict morphism is called an embedding (resp. isomorphism) of crystals. We call  $B_1$  is a subcrystal of  $B_2$ , if  $B_1$  is a subset of  $B_2$  and becomes a crystal itself by restricting the data on it from  $B_2$ .

It is well-known that the algebra  $U_q(\mathfrak{g})$  has a Hopf algebra structure. Then the tensor product of  $U_q(\mathfrak{g})$ -modules also has a  $U_q(\mathfrak{g})$ -module structure. The crystal bases have very nice properties for tensor operations. Indeed, if  $(L_i, B_i)$  is a crystal base of  $U_q(\mathfrak{g})$ -module  $M_i$  ( $i = 1, 2$ ),  $(L_1 \otimes_A L_2, B_1 \otimes B_2)$  is a crystal base of  $M_1 \otimes_{\mathbf{Q}(q)} M_2$  ([6]). Consequently, we can consider the tensor product of crystals: For crystals  $B_1$  and  $B_2$ , we define their tensor product  $B_1 \otimes B_2$  as follows:

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 : b_1 \in B_1, b_2 \in B_2\}, \quad (2.13)$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2), \quad (2.14)$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle), \quad (2.15)$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle), \quad (2.16)$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \quad (2.17)$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \quad (2.18)$$

Here  $b_1 \otimes b_2$  is just another notation for an ordered pair  $(b_1, b_2)$ , and we set  $b_1 \otimes 0 = 0 \otimes b_2 = 0 \otimes 0 = 0$ . Note that the tensor product of crystals is associative, namely, the crystals  $(B_1 \otimes B_2) \otimes B_3$  and  $B_1 \otimes (B_2 \otimes B_3)$  are isomorphic via  $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$ .

The examples of crystals below will be needed later.

**Example 2.6** (i) For  $i \in I$ , the crystal  $B_i := \{(x)_i : x \in \mathbf{Z}\}$  is defined by

$$\begin{aligned} wt((x)_i) &= x\alpha_i, & \varepsilon_i((x)_i) &= -x, & \varphi_i((x)_i) &= x, \\ \varepsilon_j((x)_i) &= -\infty, & \varphi_j((x)_i) &= -\infty & \text{for } j \neq i, \\ \tilde{e}_j(x)_i &= \delta_{i,j}(x+1)_i, & \tilde{f}_j(x)_i &= \delta_{i,j}(x-1)_i, \end{aligned}$$

(ii) Let  $R_\lambda := \{r_\lambda\}$  ( $\lambda \in P$ ) be the crystal consisting of one-element given by (see also [4]):

$$wt(r_\lambda) = \lambda, \quad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \quad \varphi_i(r_\lambda) = 0, \quad \tilde{e}_i(r_\lambda) = \tilde{f}_i(r_\lambda) = 0.$$

(iii)  $B(\lambda)$  and  $B(\infty)$  can be seen as crystals by the following way. We define the weight function  $wt : B(\lambda) \rightarrow P$  by  $wt(b) := \lambda - \sum_j \alpha_{i_j}$  for  $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \bmod qL(\lambda) \neq 0$ . We define integer-valued functions  $\varepsilon_i$  and  $\varphi_i$  on  $B(\lambda)$  by

$$\varepsilon_i(b) := \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) := \max\{k : \tilde{f}_i^k b \neq 0\}.$$

As for  $B(\infty)$ , the functions  $wt$ ,  $\varepsilon_i$  and  $\varphi_i$  are given by:  $wt(b) := -\sum_j \alpha_{i_j}$  for  $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\infty \bmod qL(\infty)$ ,

$$\varepsilon_i(b) := \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) := \varepsilon_i(b) + \langle h_i, wt(b) \rangle.$$

It is proved in [6] that the natural projection  $\pi_\lambda : U_q^-(\mathfrak{g}) \rightarrow V(\lambda)$  sends  $L(\infty)$  to  $L(\lambda)$ , and the induced map  $\hat{\pi}_\lambda : L(\infty)/qL(\infty) \rightarrow L(\lambda)/qL(\lambda)$  sends  $B(\infty)$  to  $B(\lambda) \sqcup \{0\}$ . The map  $\hat{\pi}_\lambda$  has the following properties:

$$\tilde{f}_i \circ \hat{\pi}_\lambda = \hat{\pi}_\lambda \circ \tilde{f}_i, \tag{2.19}$$

$$\tilde{e}_i \circ \hat{\pi}_\lambda = \hat{\pi}_\lambda \circ \tilde{e}_i, \text{ if } \hat{\pi}_\lambda(b) \neq 0, \tag{2.20}$$

$$\hat{\pi}_\lambda : B(\infty) \setminus \{\hat{\pi}_\lambda^{-1}(0)\} \rightarrow B(\lambda) \text{ is bijective.} \tag{2.21}$$

Although the map  $\hat{\pi}_\lambda$  has such nice properties, it is not a strict morphism of crystals. For instance, it does not preserve weights or does not necessarily commute with the action of  $\tilde{e}_i$  as in (2.20). We shall introduce a new strict morphism by modifying the map  $\hat{\pi}_\lambda$  in 3.2.

### 3 Polyhedral Realizations of Crystal Bases

#### 3.1 Polyhedral Realization of $B(\infty)$

In this subsection, we recall the results in [16].

We define a  $\mathbf{Q}(q)$ -algebra anti-automorphism  $*$  of  $U_q(\mathfrak{g})$  by:  $e_i^* = e_i$ ,  $f_i^* = f_i$ ,  $(q^h)^* = q^{-h}$ . This anti-automorphism has the properties (see [7]):

$$L(\infty)^* = L(\infty) \text{ and } B(\infty)^* = B(\infty). \tag{3.1}$$

Then we can define  $\varepsilon_i^*(b) := \varepsilon_i(b^*)$  and  $\varphi_i^*(b) := \varphi_i(b^*)$ .

Consider the additive group

$$\mathbf{Z}^\infty := \{(\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbf{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\}; \quad (3.2)$$

we will denote by  $\mathbf{Z}_{\geq 0}^\infty \subset \mathbf{Z}^\infty$  the subsemigroup of nonnegative sequences. To the rest of this section, we fix an infinite sequence of indices  $\iota = \cdots, i_k, \cdots, i_2, i_1$  from  $I$  such that

$$\#\{k : i_k = i\} = \infty \text{ for any } i \in I. \quad (3.3)$$

*Remark.* In [16][17], we assume the condition  $i_k \neq i_{k+1}$  for any  $k$ . Note that the condition  $i_k \neq i_{k+1}$  is necessary for removing some redundant components (see Lemma 4.3 below) not for the existence of the following embedding. So, without the condition  $i_k \neq i_{k+1}$  all the results in [16],[17] are also valid here. We can associate to  $\iota$  a crystal structure on  $\mathbf{Z}^\infty$  and denote it by  $\mathbf{Z}_\iota^\infty$  ([16, 2.4]).

**Proposition 3.1** ([7], See also [16]) *There is a unique embedding of crystals*

$$\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}_{\geq 0}^\infty \subset \mathbf{Z}_\iota^\infty, \quad (3.4)$$

such that  $\Psi_\iota(u_\infty) = (\cdots, 0, \cdots, 0, 0)$ .

We call this the *Kashiwara embedding* which is derived by iterating the following type of embeddings ([7]):

- (i) For any  $i \in I$ , there is a unique embedding of crystals

$$\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i, \quad (3.5)$$

such that  $\Psi_i(u_\infty) = u_\infty \otimes (0)_i$ .

- (ii) For any  $b \in B(\infty)$ , we can write uniquely  $\Psi_i(b) = b' \otimes \tilde{f}_i^m(0)_i$  where  $m = \varepsilon_i^*(b)$ .

Let us see the polyhedral realization for  $B(\infty)$ . Consider the infinite dimensional vector space

$$\mathbf{Q}^\infty := \{\vec{x} = (\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbf{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$

and its dual space  $(\mathbf{Q}^\infty)^* := \text{Hom}(\mathbf{Q}^\infty, \mathbf{Q})$ . We will write a linear form  $\varphi \in (\mathbf{Q}^\infty)^*$  as  $\varphi(\vec{x}) = \sum_{k \geq 1} \varphi_k x_k$  ( $\varphi_j \in \mathbf{Q}$ ).

For the fixed infinite sequence  $\iota = (i_k)$  we set  $k^{(+)} := \min\{l : l > k \text{ and } i_k = i_l\}$  and  $k^{(-)} := \max\{l : l < k \text{ and } i_k = i_l\}$  if it exists, or  $k^{(-)} = 0$  otherwise. We set for  $\vec{x} \in \mathbf{Q}^\infty$ ,  $\beta_0(\vec{x}) = 0$  and

$$\beta_k(\vec{x}) := x_k + \sum_{k < j < k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}} \quad (k \geq 1). \quad (3.6)$$

We define a piecewise-linear operator  $S_k = S_{k,\iota}$  on  $(\mathbf{Q}^\infty)^*$  by

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta_{k^{(-)}} & \text{if } \varphi_k \leq 0. \end{cases} \quad (3.7)$$

Here we set

$$\begin{aligned}\Xi_\iota &:= \{S_{j_l} \cdots S_{j_2} S_{j_1} x_{j_0} \mid l \geq 0, j_0, j_1, \dots, j_l \geq 1\}, \\ \Sigma_\iota &:= \{\vec{x} \in \mathbf{Z}^\infty \subset \mathbf{Q}^\infty \mid \varphi(\vec{x}) \geq 0 \text{ for any } \varphi \in \Xi_\iota\}.\end{aligned}$$

Note that in the definition of  $\Xi_\iota$  the symbol  $x_{j_0}$  is considered as an element in  $(\mathbf{Q}^\infty)^*$ , which is a function taking the corresponding coordinate. We impose on  $\iota$  the following positivity assumption:

$$\text{if } k^{(-)} = 0 \text{ then } \varphi_k \geq 0 \text{ for any } \varphi \in \Xi_\iota \text{ } (\varphi(\vec{x}) = \sum_k \varphi_k x_k).$$

**Theorem 3.2 ([16])** *Let  $\iota$  be a sequence of indices satisfying (3.3) and the positivity assumption, and  $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}_\iota^\infty$  be the Kashiwara embedding associated with  $\iota$ . Then we have  $\text{Im}(\Psi_\iota)(\cong B(\infty)) = \Sigma_\iota$ .*

*Remark.* We shall see the example of the sequence  $\iota$  which does not satisfy the positivity assumption in the end of this section.

### 3.2 Polyhedral Realization of $B(\lambda)$

We review the result in [17]. In the rest of this section,  $\lambda$  is supposed to be a dominant integral weight. The whole story below is similar to the one of  $B(\infty)$ . The essentially different point is how to deal with the data of highest weight. For the purpose, the crystal  $R_\lambda$  plays a important role, which is defined in Example 2.6 (ii). We shall introduce a new strict morphism of crystals by modifying the map  $\hat{\pi}_\lambda$ .

Consider the crystal  $B(\infty) \otimes R_\lambda$  and define the map

$$\Phi_\lambda : (B(\infty) \otimes R_\lambda) \sqcup \{0\} \longrightarrow B(\lambda) \sqcup \{0\}, \quad (3.8)$$

by  $\Phi_\lambda(0) = 0$  and  $\Phi_\lambda(b \otimes r_\lambda) = \hat{\pi}_\lambda(b)$  for  $b \in B(\infty)$ . We set

$$\tilde{B}(\lambda) := \{b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \Phi_\lambda(b \otimes r_\lambda) \neq 0\}.$$

**Theorem 3.3 ([17])** (i) *The map  $\Phi_\lambda$  becomes a surjective strict morphism of crystals  $B(\infty) \otimes R_\lambda \longrightarrow B(\lambda)$ .*

(ii)  *$\tilde{B}(\lambda)$  is a subcrystal of  $B(\infty) \otimes R_\lambda$ , and  $\Phi_\lambda$  induces the isomorphism of crystals  $\tilde{B}(\lambda) \xrightarrow{\sim} B(\lambda)$ .*

Let us denote  $\mathbf{Z}_\iota^\infty \otimes R_\lambda$  by  $\mathbf{Z}_\iota^\infty[\lambda]$ . Here note that since the crystal  $R_\lambda$  has only one element, as a set we can identify  $\mathbf{Z}_\iota^\infty[\lambda]$  with  $\mathbf{Z}_\iota^\infty$  but their crystal structures are different. By Theorem 3.3, we have the strict embedding of crystals (see also [4]):

$$\Omega_\lambda : B(\lambda)(\cong \tilde{B}(\lambda)) \hookrightarrow B(\infty) \otimes R_\lambda.$$

Combining  $\Omega_\lambda$  and the Kashiwara embedding  $\Psi_\iota$ , we obtain the following:

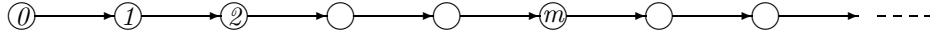
**Theorem 3.4** ([17]) *There exists the unique strict embedding of crystals*

$$\Psi_t^{(\lambda)} : B(\lambda) \xrightarrow{\Omega_\lambda} B(\infty) \otimes R_\lambda \xrightarrow{\Psi_t \otimes \text{id}} \mathbf{Z}_t^\infty \otimes R_\lambda =: \mathbf{Z}_t^\infty[\lambda], \quad (3.9)$$

such that  $\Psi_t^{(\lambda)}(u_\lambda) = (\cdots, 0, 0, 0) \otimes r_\lambda$ .

In the following example, we will see how the crystal  $R_\lambda$  works by tensoring with  $B(\infty)$ .

**Example 3.5** *Let us see the simplest example  $\mathfrak{g} = \mathfrak{sl}_2$ -case. Let  $u_\infty$  be the highest weight vector in  $B(\infty)$ . Then we have  $B(\infty) = \{\tilde{f}^n u_\infty\}$ . The crystal graph of  $B(\infty)$  is as follows:*

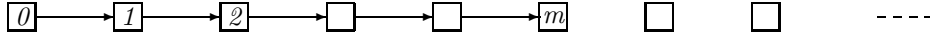


where  $\textcircled{x} = \tilde{f}^x u_\infty$ .

Next, let us see the crystal graph of  $B(\infty) \otimes R_m$  ( $m \in \mathbf{Z}_{\geq 0}$ ), where  $R_m$  is the crystal as in Example 2.6 (ii) with  $\lambda = m$ . We know that  $\varphi(\tilde{f}^n u_\infty) = -n$  and  $\varepsilon(r_m) = -m$ . Then, by (2.18) we have

$$\tilde{f}(\tilde{f}^n u_\infty \otimes r_m) = \begin{cases} \tilde{f}^{n+1} u_\infty \otimes r_m & \text{if } n < m, \\ \tilde{f}^n u_\infty \otimes \tilde{f}(r_m) = 0 & \text{if } n \geq m. \end{cases}$$

Thus, the crystal graph of  $B(\infty) \otimes R_m$  is:



where  $\boxed{x} = \tilde{f}^x u_\infty \otimes r_m$ . The connected component including  $\boxed{0} = u_\infty \otimes r_m$  is isomorphic to the crystal  $B(m)$  associated with the  $m+1$ -dimensional irreducible module  $V(m)$ . The polyhedral realization below gives us the method how to remove the vectors excluded from  $B(\lambda)$  (in this case, the vectors  $\{\boxed{x} \mid x > m\}$ ).

We shall give an explicit crystal structure of  $\mathbf{Z}_t^\infty[\lambda]$  following to [17], which is derived by using the explicit formula (2.14)–(2.18) repeatedly. Fix a sequence of indices  $\iota := (i_k)_{k \geq 1}$  satisfying the condition (3.3) and a weight  $\lambda \in P$ . (Here we do not necessarily assume that  $\lambda$  is dominant.) As we stated before, we can identify  $\mathbf{Z}^\infty$  with  $\mathbf{Z}_t^\infty[\lambda]$  as a set. Thus  $\mathbf{Z}_t^\infty[\lambda]$  can be regarded as a subset of  $\mathbf{Q}^\infty$ , and then we denote an element in  $\mathbf{Z}_t^\infty[\lambda]$  by  $\vec{x} = (\cdots, x_k, \cdots, x_2, x_1)$ . For  $\vec{x} = (\cdots, x_k, \cdots, x_2, x_1) \in \mathbf{Q}^\infty$  we define the linear functions

$$\sigma_k(\vec{x}) := x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j \quad (k \geq 1), \quad (3.10)$$

$$\sigma_0^{(i)}(\vec{x}) := -\langle h_i, \lambda \rangle + \sum_{j \geq 1} \langle h_i, \alpha_{i_j} \rangle x_j \quad (i \in I). \quad (3.11)$$

Here note that since  $x_j = 0$  for  $j \gg 0$  on  $\mathbf{Q}^\infty$ , the functions  $\sigma_k$  and  $\sigma_0^{(i)}$  are well-defined. For  $\vec{x} \in \mathbf{Q}^\infty$  let  $\sigma^{(i)}(\vec{x}) := \max_{k: i_k = i} \sigma_k(\vec{x})$ , and

$$M^{(i)} = M^{(i)}(\vec{x}) := \{k : i_k = i, \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x})\}. \quad (3.12)$$



Note that  $\sigma^{(i)}(\vec{x}) \geq 0$ , and that  $M^{(i)} = M^{(i)}(\vec{x})$  is a finite set if and only if  $\sigma^{(i)}(\vec{x}) > 0$ . Now we define the maps  $\tilde{e}_i : \mathbf{Z}_\iota^\infty[\lambda] \sqcup \{0\} \longrightarrow \mathbf{Z}_\iota^\infty[\lambda] \sqcup \{0\}$  and  $\tilde{f}_i : \mathbf{Z}_\iota^\infty[\lambda] \sqcup \{0\} \longrightarrow \mathbf{Z}_\iota^\infty[\lambda] \sqcup \{0\}$  by setting  $\tilde{e}_i(0) = f_i(0) = 0$ , and

$$(\tilde{f}_i(\vec{x}))_k = x_k + \delta_{k, \min M^{(i)}} \text{ if } \sigma^{(i)}(\vec{x}) > \sigma_0^{(i)}(\vec{x}); \text{ otherwise } \tilde{f}_i(\vec{x}) = 0, \quad (3.13)$$

$$(\tilde{e}_i(\vec{x}))_k = x_k - \delta_{k, \max M^{(i)}} \text{ if } \sigma^{(i)}(\vec{x}) > 0 \text{ and } \sigma^{(i)}(\vec{x}) \geq \sigma_0^{(i)}(\vec{x}); \text{ otherwise } \tilde{e}_i(\vec{x}) = 0, \quad (3.14)$$

where  $\delta_{i,j}$  is the Kronecker's delta. Here note that  $\tilde{e}_i$  and  $\tilde{f}_i$  act on the (infinite) tensor product of crystals and they act on just one component in it. The functions  $\sigma_k$  and  $\sigma_0^{(i)}$  given above determine which component is changed by  $\tilde{e}_i$  and  $\tilde{f}_i$ .

We also define the weight function and the functions  $\varepsilon_i$  and  $\varphi_i$  on  $\mathbf{Z}^\infty[\lambda]$  by

$$\begin{aligned} wt(\vec{x}) &:= \lambda - \sum_{j=1}^\infty x_j \alpha_{i_j}, \quad \varepsilon_i(\vec{x}) := \max(\sigma^{(i)}(\vec{x}), \sigma_0^{(i)}(\vec{x})) \\ \varphi_i(\vec{x}) &:= \langle h_i, wt(\vec{x}) \rangle + \varepsilon_i(\vec{x}). \end{aligned} \quad (3.15)$$

Now we obtain the explicit crystal structure of  $\mathbf{Z}_\iota^\infty[\lambda]$ . Note that, in general, the subset  $\mathbf{Z}_{\geq 0}^\infty[\lambda]$  is not a subcrystal of  $\mathbf{Z}_\iota^\infty[\lambda]$  since it is not stable under the action of  $\tilde{e}_i$ 's.

We fix a sequence of indices  $\iota$  satisfying (3.3) and take a dominant integral weight  $\lambda \in P_+$ . For  $k \geq 1$  let  $k^{(\pm)}$  be the ones in 2.4. Let  $\beta_k^{(\pm)}$  be linear functions given by

$$\beta_k^{(+)}(\vec{x}) = \sigma_k(\vec{x}) - \sigma_{k^{(+)}}(\vec{x}) = x_k + \sum_{k < j < k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}} \quad (3.16)$$

$$\begin{aligned} &\beta_k^{(-)}(\vec{x}) \\ &= \begin{cases} \sigma_{k^{(-)}}(\vec{x}) - \sigma_k(\vec{x}) = x_{k^{(-)}} + \sum_{k^{(-)} < j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_k & \text{if } k^{(-)} > 0, \\ \sigma_0^{(i_k)}(\vec{x}) - \sigma_k(\vec{x}) = -\langle h_{i_k}, \lambda \rangle + \sum_{1 \leq j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_k & \text{if } k^{(-)} = 0, \end{cases} \end{aligned} \quad (3.17)$$

where the functions  $\sigma_k$  and  $\sigma_0^{(i)}$  are defined by (3.10) and (3.11). Here note that  $\beta_k^{(+)} = \beta_k$  and  $\beta_k^{(-)} = \beta_{k^{(-)}}$  if  $k^{(-)} > 0$ .

Using this notation, for every  $k \geq 1$ , we define an operator  $\widehat{S}_k = \widehat{S}_{k, \iota}$  for a linear function  $\varphi(\vec{x}) = c + \sum_{k \geq 1} \varphi_k x_k$  ( $c, \varphi_k \in \mathbf{Q}$ ) on  $\mathbf{Q}^\infty$  by:

$$\widehat{S}_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k^{(+)} & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta_k^{(-)} & \text{if } \varphi_k \leq 0. \end{cases} \quad (3.18)$$

An easy check shows that  $(\widehat{S}_k)^2 = \widehat{S}_k$ .

For the fixed sequence  $\iota = (i_k)$ , we define  $\iota^{(i)}$  ( $i \in I$ ) to be the index  $k \geq 1$  such that  $i_k = i$  and  $k^{(-)} = 0$ . Note that such  $k$  is uniquely determined. Simply,  $\iota^{(i)}$  is the first  $k$  such that  $i_k = i$ .

Here for  $\lambda \in P_+$  and  $i \in I$  we set

$$\lambda^{(i)}(\vec{x}) := -\beta_{\iota^{(i)}}^{(-)}(\vec{x}) = \langle h_i, \lambda \rangle - \sum_{1 \leq j < \iota^{(i)}} \langle h_i, \alpha_{i_j} \rangle x_j - x_{\iota^{(i)}}. \quad (3.19)$$

For  $\iota$  and a dominant integral weight  $\lambda$ , let  $\Xi_\iota[\lambda]$  be the set of all linear functions generated by  $\widehat{S}_k = \widehat{S}_{k,\iota}$  on the functions  $x_j$  ( $j \geq 1$ ) and  $\lambda^{(i)}$  ( $i \in I$ ), namely,

$$\begin{aligned} \Xi_\iota[\lambda] := & \{\widehat{S}_{j_l} \cdots \widehat{S}_{j_1} x_{j_0} : l \geq 0, j_0, \dots, j_l \geq 1\} \\ & \cup \{\widehat{S}_{j_k} \cdots \widehat{S}_{j_1} \lambda^{(i)}(\vec{x}) : k \geq 0, i \in I, j_1, \dots, j_k \geq 1\}. \end{aligned} \quad (3.20)$$

Now we set

$$\Sigma_\iota[\lambda] := \{\vec{x} \in \mathbf{Z}_\iota^\infty[\lambda] \subset \mathbf{Q}^\infty : \varphi(\vec{x}) \geq 0 \text{ for any } \varphi \in \Xi_\iota[\lambda]\}. \quad (3.21)$$

Here note that in general, it is possible that the set  $\Sigma_\iota[\lambda]$  is empty. This never occurs in the case of  $B(\infty)$ . In the case of  $B(\lambda)$ , any element in  $\Xi_\iota$  has trivial constant term, but in the base of  $B(\lambda)$ , some element in  $\Xi_\iota[\lambda]$  may have a negative constant term. But here the case  $\Sigma_\iota[\lambda] \ni \vec{0} = (\dots, 0, 0)$  is only considered in the sequel and in this case, a pair  $(\iota, \lambda)$  is called *ample*. The assumption 'ample' requires that the element corresponding to the highest weight vector is contained in  $\Sigma_\iota[\lambda]$ .

**Theorem 3.6 ([17])** *Suppose that  $(\iota, \lambda)$  is ample. Let  $\Psi_\iota^{(\lambda)} : B(\lambda) \hookrightarrow \mathbf{Z}_\iota^\infty[\lambda]$  be the embedding as in (3.9). Then the image  $\text{Im}(\Psi_\iota^{(\lambda)}) (\cong B(\lambda))$  is equal to  $\Sigma_\iota[\lambda]$ .*

### 3.3 Rank 2 case

We apply Theorem 3.6 to the case of the Kac-Moody algebras of rank 2. The setting here is the same as [17]. Without loss of generality, we can and will assume that  $I = \{1, 2\}$ , and  $\iota = (\dots, 2, 1, 2, 1)$ . The Cartan data is given by:

$$\langle h_1, \alpha_1 \rangle = \langle h_2, \alpha_2 \rangle = 2, \quad \langle h_1, \alpha_2 \rangle = -c_1, \quad \langle h_2, \alpha_1 \rangle = -c_2.$$

Here we either have  $c_1 = c_2 = 0$ , or both  $c_1$  and  $c_2$  are positive integers. We set  $X = c_1 c_2 - 2$ , and define the integer sequence  $a_l = a_l(c_1, c_2)$  for  $l \geq 0$  by setting  $a_0 = 0$ ,  $a_1 = 1$  and, for  $k \geq 1$ ,

$$a_{2k} = c_1 P_{k-1}(X), \quad a_{2k+1} = P_k(X) + P_{k-1}(X), \quad (3.22)$$

where the  $P_k(X)$  are *Chebyshev polynomials* given by the following generating function:

$$\sum_{k \geq 0} P_k(X) z^k = (1 - Xz + z^2)^{-1}. \quad (3.23)$$

Here define  $a'_l(c_1, c_2) := a_l(c_2, c_1)$ . The several first Chebyshev polynomials and terms  $a_l$  are given by

$$P_0(X) = 1, \quad P_1(X) = X, \quad P_2(X) = X^2 - 1, \quad P_3(X) = X^3 - 2X,$$

$$a_2 = c_1, \quad a_3 = c_1 c_2 - 1, \quad a_4 = c_1(c_1 c_2 - 2),$$

$$a_5 = (c_1 c_2 - 1)(c_1 c_2 - 2) - 1, \quad a_6 = c_1(c_1 c_2 - 1)(c_1 c_2 - 3),$$

$$a_7 = c_1 c_2(c_1 c_2 - 2)(c_1 c_2 - 3) - 1.$$

Let  $l_{\max} = l_{\max}(c_1, c_2)$  be the minimal index  $l$  such that  $a_{l+1} < 0$  (if  $a_l \geq 0$  for all  $l \geq 0$ , then we set  $l_{\max} = +\infty$ ). By inspection, if  $c_1 c_2 = 0$  (resp. 1, 2, 3) then  $l_{\max} = 2$  (resp. 3, 4, 6). Furthermore, if  $c_1 c_2 \leq 3$  then  $a_{l_{\max}} = 0$  and  $a_l > 0$  for  $1 \leq l < l_{\max}$ . On the other hand, if  $c_1 c_2 \geq 4$ , i.e.,  $X \geq 2$ , it is easy to see from (3.23) that  $P_k(X) > 0$  for  $k \geq 0$ , hence  $a_l > 0$  for  $l \geq 1$ ; in particular, in this case  $l_{\max} = +\infty$ .

**Theorem 3.7 ([17])** *In the rank 2 case, for a dominant integral weight  $\lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2$  ( $\lambda_1, \lambda_2 \in \mathbf{Z}_{\geq 0}$ ) the image of the embedding  $\Psi_t^{(\lambda)}$  is given by*

$$\text{Im}(\Psi_t^{(\lambda)}) = \left\{ (\cdots, x_2, x_1) \in \mathbf{Z}_{\geq 0}^\infty : \begin{array}{l} x_k = 0 \text{ for } k > l_{\max}, \lambda_1 \geq x_1, \\ a_l x_l - a_{l-1} x_{l+1} \geq 0, \\ \lambda_2 + a'_{l+1} x_l - a'_l x_{l+1} \geq 0, \\ \text{for } 1 \leq l < l_{\max} \end{array} \right\}. \quad (3.24)$$

Note that the cases when  $l_{\max} < +\infty$ , or equivalently, the image  $\text{Im}(\Psi_t)$  is contained in a lattice of finite rank, just correspond to the Lie algebras  $\mathfrak{g} = A_1 \times A_1, A_2, B_2$  or  $C_2, G_2$ .

In conclusion of this section, we illustrate Theorem 3.7 by the example when  $c_1 = c_2 = 2$ , i.e.,  $\mathfrak{g}$  is the affine Lie algebra of type  $A_1^{(1)}$ . In this case,  $X = c_1 c_2 - 2 = 2$ . It follows at once from (3.23) that  $P_k(2) = k + 1$ ; hence, (3.22) gives  $a_l = l$  for  $l \geq 0$ . We see that for type  $A_1^{(1)}$ ,

$$B(\lambda) \cong \text{Im}(\Psi_t^{(\lambda)}) = \{(\cdots, x_2, x_1) \in \mathbf{Z}_{\geq 0}^\infty : \begin{array}{l} l x_l - (l-1) x_{l+1} \geq 0, \lambda_1 \geq x_1 \text{ and} \\ \lambda_2 + (l+1) x_l - l x_{l+1} \geq 0 \text{ for } l \geq 1 \end{array} \},$$

### 3.4 $A_n$ -case

Next, we shall apply Theorem 3.6 to the case when  $\mathfrak{g}$  is of type  $A_n$ . Let us identify the index set  $I$  with  $[1, n] := \{1, 2, \dots, n\}$  in the standard way; thus, the Cartan matrix  $(a_{i,j} = \langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$  is given by  $a_{i,i} = 2$ ,  $a_{i,j} = -1$  for  $|i - j| = 1$ , and  $a_{i,j} = 0$  otherwise. As the infinite sequence  $\iota$  let us take the following periodic sequence

$$\iota = \cdots, \underbrace{n, \dots, 2, 1}, \cdots, \underbrace{n, \dots, 2, 1}, \underbrace{n, \dots, 2, 1}.$$

Following to [16, 17], we shall change the indexing set for  $\mathbf{Z}_t^\infty$  from  $\mathbf{Z}_{\geq 1}$  to  $\mathbf{Z}_{\geq 1} \times [1, n]$ , which is given by the bijection  $\mathbf{Z}_{\geq 1} \times [1, n] \rightarrow \mathbf{Z}_{\geq 1} ((j; i) \mapsto (j-1)n + i)$ . According to this, we will write an element  $\vec{x} \in \mathbf{Z}^\infty$  as a doubly-indexed family  $(x_{j;i})_{j \geq 1, i \in [1, n]}$ . We will adopt the convention that  $x_{j;i} = 0$  unless  $j \geq 1$  and  $i \in [1, n]$ ; in particular,  $x_{j;0} = x_{j;n+1} = 0$  for all  $j$ .

**Theorem 3.8 ([17])** *Let  $\lambda = \sum_{1 \leq i \leq n} \lambda_i \Lambda_i$  ( $\lambda_i \in \mathbf{Z}_{\geq 0}$ ) be a dominant integral weight. In the above notation, the image  $\text{Im}(\Psi_t^{(\lambda)})$  is the set of all integer families  $(x_{j;i})$  such that*

$$x_{1;i} \geq x_{2;i-1} \geq \cdots \geq x_{i;1} \geq 0 \text{ for } 1 \leq i \leq n \quad (3.25)$$

$$x_{j;i} = 0 \text{ for } i + j > n + 1, \quad (3.26)$$

$$\lambda_i \geq x_{j,i-j+1} - x_{j,i-j} \text{ for } 1 \leq j \leq i \leq n. \quad (3.27)$$

We give the example which does not satisfy the positivity assumption.

**Example 3.9 ([17])** We consider the case  $\mathfrak{g} = A_3$  and take the sequence  $\iota = \dots 212321$ , where we do not need the explicit form of “ $\dots$ ” in  $\iota$ . For simplicity, we write  $\vec{x} = (\dots, x_2, x_1)$  for an element  $\vec{x} \in \mathbf{Z}_\iota^\infty$ . In this setting, we have  $\beta_1 = x_1 - x_2 - x_4 + x_5$ ,  $\beta_2 = x_2 - x_3 + x_4$  and  $5^{(-)} = 1$ . Then  $S_1(x_1) = x_1 - \beta_1 = x_2 + x_4 - x_5$ ,  $S_2S_1(x_1) = x_2 + x_4 - x_5 - \beta_2 = x_3 - x_5$  and  $S_5S_2S_1(x_1) = x_3 - x_5 + \beta_1 = x_1 - x_2 + x_3 - x_4$ . Thus we see  $S_5S_2S_1(x_1)$  has the negative coefficient for  $x_2$ , which breaks the positivity assumption. Furthermore, this case is not ample. Fix  $\lambda \in P_+$  with  $\langle h_2, \lambda \rangle > 0$ . Since  $\beta_2^{(-)} = -\langle h_2, \lambda \rangle + x_2 - x_1$  and  $\widehat{S}_5\widehat{S}_2\widehat{S}_1(x_1) = S_5S_2S_1(x_1)$ , we have

$$\widehat{S}_2\widehat{S}_5\widehat{S}_2\widehat{S}_1(x_1) = x_1 - x_2 + x_3 - x_4 + \beta_2^{(-)} = -\langle h_2, \lambda \rangle + x_3 - x_4,$$

which implies  $\vec{0} = (\dots, 0, 0) \notin \Sigma_\iota[\lambda]$  since  $\langle h_2, \lambda \rangle > 0$ .

## 4 Braid-type isomorphisms and its application

### 4.1 Braid-type isomorphisms

In this subsection we shall give the “braid-type isomorphisms” of crystals.

Let  $I$  be the finite index set and  $B_i$  and  $B_j$  ( $i, j \in I$ ) be the crystals as in Example 2.6 (i) with the condition

$$c_{ij} := \langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle \leq 3. \quad (4.1)$$

Set  $c_1 := -\langle h_i, \alpha_j \rangle$  and  $c_2 := -\langle h_j, \alpha_i \rangle$ . In the sequel, for  $x \in \mathbf{Q}$  we set

$$x_+ := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

**Proposition 4.1** (i) Under the condition (4.1) there exist the following type of isomorphisms of crystals  $\phi_{ij}^{(k)}$  ( $k = 0, 1, 2, 3$ ):

(a) If  $c_{ij} = 0$ ,

$$\phi_{ij}^{(0)} : B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i,$$

$$\text{whetre } \phi_{ij}^{(0)}((x)_i \otimes (y)_j) = (y)_j \otimes (x)_i.$$

(b) If  $c_{ij} = 1$ ,

$$\phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j,$$

where

$$\phi_{ij}^{(1)}((x)_i \otimes (y)_j \otimes (z)_i) = (z + (-x + y - z)_+)_j \otimes (x + z)_i \otimes (y - z - (-x + y - z)_+)_j.$$

(c) If  $c_{ij} = 2$ ,

$$\phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i,$$

where  $\phi_{ij}^{(2)}$  is given by the following: for  $(x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j$  we set  $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i := \phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)$ .

$$X = w + (-c_2x + y - w + c_2(x - c_1y + z)_+)_+, \quad (4.2)$$

$$Y = x + c_1w + (-x + z - c_1w + (x - c_1y + z)_+)_+, \quad (4.3)$$

$$Z = y - (-c_2x + y - w + c_2(x - c_1y + z)_+)_+, \quad (4.4)$$

$$W = z - c_1w - (-x + z - c_1w + (x - c_1y + z)_+)_+. \quad (4.5)$$

(d) If  $c_{ij} = 3$ ,

$$\phi_{ij}^{(3)} : B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i,$$

where it is defined by the following: for  $(x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j$  we set  $A := -x + c_1y - z$ ,  $B := -y + c_2z - u$ ,  $C := -z + c_1u - v$  and  $D := -u + c_2v - w$ . Then  $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (U)_i \otimes (V)_j \otimes (W)_i := \phi_{ij}^{(3)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j)$  is given by

$$X = w + (D + (c_2C + (2B + c_2A_+)_+)_+)_+, \quad (4.6)$$

$$Y = x + c_1w + (c_1D + (3C + (2c_1B + 2A_+)_+)_+)_+, \quad (4.7)$$

$$Z = y + u + w - X - V, \quad (4.8)$$

$$U = x + z + v - Y - W, \quad (4.9)$$

$$V = u - w - (2D + (2c_2C + (3B + c_2A_+)_+)_+)_+, \quad (4.10)$$

$$W = v - c_1w - (c_1D + (2C + (c_1B + A_+)_+)_+)_+. \quad (4.11)$$

(ii) For  $k = 0, 1, 2, 3$ , we have  $\phi_{ji}^{(k)} \circ \phi_{ij}^{(k)} = \text{id}$ .

We call  $\phi_{ij}^{(k)}$  a *braid-type isomorphism*.

In [14], Littelmann gave the similar formula to the one in Proposition 4.1 for the cone realized by his path model (he omitted the complete form for the  $G_2$ -case.). The different points are as follows: (1) he obtained it by considering the actions of the Weyl groups but we got the formula by requiring only that it should be an isomorphism of crystals, and (2) our isomorphism is for the crystal  $B_i \otimes B_j \otimes \cdots$ , which contains the cone as a subset.

*Proof.* The bijectivity of  $\phi_{ij}^{(k)}$  follows immediately from (ii). Thus, first let us show (ii). The case (a) is trivial. In the case (b), set  $(X)_j \otimes (Y)_i \otimes (Z)_j := \phi_{ij}^{(1)}((x)_i \otimes (y)_j \otimes (z)_i)$  and  $(X')_i \otimes (Y')_j \otimes (Z')_i := \phi_{ji}^{(1)}((X)_j \otimes (Y)_i \otimes (Z)_j)$ . Then we have  $X' = Z + (-X + Y - Z)_+ = y - z - (-x + y + z)_+ + (x - y + z)_+ = y - z + (x - y + z) = x$ ,  $Y' = X + Z = y$  and  $Z' = Y - Z - (-X + Y - Z)_+ = x - y + 2z + (-x + y - z)_+ - (x - y + z)_+ = x - y + 2z - (x - y + z) = z$ . Here we use the formula:  $x_+ - (-x)_+ = x$ . Next, let us see the case (c). We shall divide the cases by the signs of the following four values:  $P := x - c_1y + z$ ,  $Q := -c_2x + y - w$ ,  $R := -x + z - c_1w$  and  $S := -y + c_2z - w$ . Those have the relations:  $P + c_1Q = R$ ,  $c_2P + Q = S$ ,  $Q + S = c_2R$  and  $-P + c_1S = R$ . Set  $(\alpha)_i \otimes (\beta)_j \otimes (\gamma)_j \otimes (\delta)_j = \phi_{ji}^{(2)} \circ \phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)$ . Here note that  $\phi_{ji}^{(2)}$  is obtained by exchanging  $c_1$  and  $c_2$  in  $\phi_{ij}^{(2)}$ . We shall see the case  $P \leq 0$  and  $Q \leq 0$ .

We have  $\phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = (w)_j \otimes (x + c_1 w)_i \otimes (y)_i \otimes (z - c_1 w)_i$  and then  $\alpha = z - c_1 w + (x - z + c_1 w + c_1 Q_+)_+ = z - c_1 w + (x - z + c_1 w)_+ = x$  since  $x - z + c_1 w = -R = -P - c_1 Q \geq 0$ . We also have  $\beta = c_2 x - w + (-c_2 P - Q + Q_+)_+ = c_2 x - w + (y - c_2 z + w) = y$ . Since  $\alpha + \gamma = x + z$  and  $\beta + \delta = y + w$ , we also have  $\gamma = z$  and  $\delta = w$ . By arguing similarly for the other cases we can get  $\alpha = x$ ,  $\beta = y$ ,  $\gamma = z$  and  $\delta = w$ .

The proof for the case of  $\phi_{ij}^{(3)}$  is not difficult but quite complicated. It would be easier to show if we adopt another expressions for  $X, \dots, W$ :

$$\begin{aligned} X &= \max(-c_2 x + y, -2y + c_2 z, -c_2 z + 2u, -u + c_2 v, w), \\ Y &= \max(-x + z, x - 2c_1 y + 3z, x - 3z + 2c_1 u, x - c_1 u + 3v, x + c_1 w), \\ Z &= y + u + w - X - V, \\ U &= x + z + v - Y - W, \\ V &= \min(c_2 x + w, 3y - c_2 z + w, 2c_2 z - 3u + w, 3u - 2c_2 v + w, u - w), \\ W &= \min(x, c_1 y - z, 2z - c_1 u, c_1 u - 2v, v - c_1 w). \end{aligned}$$

Set  $(X')_i \otimes (Y')_j \otimes (Z')_i \otimes (U')_j \otimes (V')_i \otimes (W')_j := \phi_{ji}^{(3)}((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (U)_i \otimes (V)_j \otimes (W)_i)$ . To see (ii) in this case, we may show  $(X', Y', Z', U', V', W') = (x, y, z, u, v, w)$ . For example, in the following case we shall show  $W' = w$ : suppose that  $2B + c_2 A_+ \geq 0 > 3B + c_2 A_+, c_2 C + 2B + c_2 A_+ \geq 0$  and  $D + c_2 C + 2B + c_2 A_+ < 0$ . In this case, we can show

$$X = w, \quad c_2 Y - Z \geq X, \quad 2Z - c_2 U \geq X, \quad c_2 U - 2V \geq X, \quad V - c_2 W \geq X. \quad (4.12)$$

Since  $W' = \min(X, c_2 Y - Z, 2Z - c_2 U, c_2 U - 2V, V - c_2 W)$ , we have  $W' = X = w$  by (4.12). What we should show here is same as the case of  $\phi_{ij}^{(2)}$ , so other cases are remained to the readers.

Let us show (i). The bijectivity is obtained by (ii). So we shall see that the map  $\phi_{ij}^{(k)}$  is a strict morphism of crystals. We should check the following:

- (1) The map  $\phi_{ij}^{(k)}$  preserves the data  $wt, \varepsilon_i$ , and  $\varphi_i$  for  $i \in I$ .
- (2) The map  $\phi_{ij}^{(k)}$  commutes all  $\tilde{e}_i$  and  $\tilde{f}_i$ .

Let us see (1) for the case (c). We have  $wt((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = (x + z)\alpha_i + (y + w)\alpha_j$  and  $wt(\phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)) = (Y + W)\alpha_i + (X + Z)\alpha_j$ . By the explicit forms of  $X, Y, Z, W$  in (4.2)–(4.5),  $X + Z = y + w$  and  $Y + W = x + z$ . Hence, we have  $wt(v) = wt(\phi_{ij}^{(2)}(v))$  for  $v \in B_i \otimes B_j \otimes B_i \otimes B_j$ . Next, let us see  $\varepsilon_i$ . In the case  $c_1 = 2$  and  $c_2 = 1$ , we have

$$\varepsilon_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = -x + (-x + 2y - z)_+, \quad (4.13)$$

$$\varepsilon_i((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i) = 2X - Y + (-Y + 2Z - W)_+, \quad (4.14)$$

$$\varepsilon_j((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = x - y + (-y + z - w)_+, \quad (4.15)$$

$$\varepsilon_j((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i) = -X + (-X + Y - Z)_+. \quad (4.16)$$

If  $x - 2y + z \leq 0$ ,  $\varepsilon_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = -2x + 2y - z$ . Furthermore, if  $-x + y - w \geq 0$ , we have  $2X - Y + (-Y + 2Z - W)_+ = -2x + 2y - x - 2w - (-x +$

$z - 2w)_+ + (x - z + 2w)_+ = -3x + 2y - 2w + (x - z + 2w) = -2x + 2y - z$ . If  $-x + y - w \leq 0$ , we have  $2X - Y + (-Y + 2Z - W)_+ = 2w - x - 2w - (-x + z - 2w)_+ + (-x + 2y - z)_+ = -2x + 2y - z$  since  $-x + z - 2w = 2(-x + y - w) + (x - 2y + z) \leq 0$ . Thus, we have  $\varepsilon_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = \varepsilon_i((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i)$  in the case  $x - 2y + z \leq 0$ . We can obtain other cases by the similar argument. Hence, the function  $\varepsilon_i$  is preserved by  $\phi_{ij}^{(2)}$ . The case of  $\varphi_i$  is trivial due to the formula  $\varphi_i(b) = \langle h_i, wt(b) \rangle + \varepsilon_i(b)$ , and the fact that the RHS of this formula is preserved by  $\phi_{ij}^{(2)}$ . The cases for  $\varepsilon_j$  and  $\varphi_j$  are obtained by the similar way. The cases for  $\varepsilon_j$  and  $\varphi_j$  are obtained by the similar way. We finished (1).

Let us show (2). For  $x, y, z, w$  let  $X, Y, Z, W$  be as in (4.2)–(4.5). We shall show the case of  $\tilde{f}_i$ . The other cases are shown similarly.

By the formula (2.18) we have

$$\begin{aligned} & \tilde{f}_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) \\ &= \begin{cases} (x-1)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j & \text{if } x - c_1y + z > 0, \\ (x)_i \otimes (y)_j \otimes (z-1)_i \otimes (w)_j & \text{if } x - c_1y + z \leq 0, \end{cases} \end{aligned} \quad (4.17)$$

$$\begin{aligned} & \tilde{f}_i((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i) \\ &= \begin{cases} (X)_j \otimes (Y-1)_i \otimes (Z)_j \otimes (W)_i & \text{if } Y - c_1Z + W > 0, \\ (X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W-1)_i & \text{if } Y - c_1Z + W \leq 0. \end{cases} \end{aligned} \quad (4.18)$$

(i) The case  $x - c_1y + z > 0$ :

In this case, we have  $\tilde{f}_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = (x-1)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j$  and,  $X = w + (-y + c_2z - w)_+$ ,  $Y = x + c_1w + c_1(-y + c_2z - w)_+$ ,  $Z = y - (-y + c_2z - w)_+$  and  $W = z - c_1w - c_1(-y + c_2z - w)_+$ . Thus,  $Y - c_1Z + W = x - c_1y + z + c_1(-y + c_2z - w)_+ \geq x - c_1y + w > 0$  and then we have  $\tilde{f}_i((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i) = (X)_j \otimes (Y-1)_i \otimes (Z)_j \otimes (W)_i$ . Set  $(X')_j \otimes (Y')_i \otimes (Z')_j \otimes (W')_i := \phi_{ij}^{(2)}((x-1)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)$ . Since  $x - c_1y + z > 0$ , we have  $x - c_1y + z - 1 \geq 0$ . Thus, by the explicit form of  $X, Y, Z$  and  $W$  we have  $X' = X$ ,  $Y' = Y - 1$ ,  $Z' = Z$  and  $W' = W$ , which implies that  $\tilde{f}_i \circ \phi_{ij}^{(2)} = \phi_{ij}^{(2)} \circ \tilde{f}_i$  in the case  $x - c_1y + z > 0$ .

(ii) The case  $x - c_1y + z \leq 0$ :

In this case,  $\tilde{f}_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j) = (x)_i \otimes (y)_j \otimes (z-1)_i \otimes (w)_j$ . We have  $X = w + (-c_2x + y - w)_+$ ,  $Y = x + c_1w + (-x + z - c_1w)_+$ ,  $Z = y - (-c_2x + y - w)_+$  and  $W = z - c_1w - (-x + z - c_1w)_+$ . Set  $(X'')_j \otimes (Y'')_i \otimes (Z'')_j \otimes (W'')_i := \phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z-1)_i \otimes (w)_j)$ . If  $-x + z - c_1w > 0$ ,  $-c_2x + y - w = \{(-x + z - c_1w) + (-x + c_1y - z)\}/c_1 > 0$  and then we have

$$X = -c_2x + y, \quad Y = z, \quad Z = c_2x + w, \quad W = x. \quad (4.19)$$

Thus,  $Y - c_1Z + W = -x + z - c_1w > 0$  and then  $\tilde{f}_i((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i) = (X)_j \otimes (Y-1)_i \otimes (Z)_j \otimes (W)_i$ . In this case, since  $x - c_1y + z - 1 < 0$  and  $-x + (z-1) - c_1w \geq 0$ , by (4.19) we have  $X'' = X$ ,  $Y'' = Y - 1$ ,  $Z'' = Z$  and  $W'' = W$ , which implies that  $\tilde{f}_i \circ \phi_{ij}^{(2)} = \phi_{ij}^{(2)} \circ \tilde{f}_i$  in this case.

If  $-x + z - c_1w \leq 0$ , we have

$$X = w + (-c_2x + y - w)_+, \quad Y = x + c_1w, \quad Z = y - (-c_2x + y - w)_+, \quad W = z - c_1w. \quad (4.20)$$

Thus, we have

$$\begin{aligned} Y - c_1 Z + W &= x - c_1 y + z + c_1(-c_2 x + y - w)_+ \\ &= \max\{x - c_1 y + z, -x + z - c_1 w\} \leq 0. \end{aligned}$$

So we have  $\tilde{f}_i((X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i) = (X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W-1)_i$ . In this case, we have still  $-x + (z-1) - c_1 w < 0$ , and then by (4.20)  $X'' = X$ ,  $Y'' = Y$ ,  $Z'' = Z$  and  $W'' = W - 1$ , which implies that  $\tilde{f}_i \phi_{ij}^{(2)} = \phi_{ij}^{(2)} \tilde{f}_i$ .

Now we have completed to show the commutativity of  $\tilde{f}_i$  and  $\phi_{ij}^{(2)}$ . The case of  $\tilde{e}_i$ ,  $\tilde{e}_j$  and  $\tilde{f}_j$  can be shown similarly. Now we know that the map  $\phi_{ij}^{(2)}$  is the strict morphism of crystals and then due to (1) and (2),  $\phi_{ij}^{(2)}$  turns out to be the isomorphism of crystals.

Next, we shall see that  $\phi_{ij}^{(3)}$  is a strict morphism of crystals. Let us show (1). It is trivial that  $\phi_{ij}^{(3)}$  preserves the weight by its explicit form. As for  $\varepsilon_i$ , we have  $\varepsilon_i((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j) = \max(-x, -x+A, -x+A+C)$ . Therefore, we consider the following three cases: (I)  $A \leq 0$  and  $A+C \leq 0$ . (II)  $A \geq 0$  and  $C \leq 0$ . (III)  $A \geq 0$  and  $A+C \geq 0$ . In each case, we have  $\varepsilon_i((x)_i \otimes \cdots \otimes (w)_j) = -x, -x+A, -x+A+C$  respectively. And by inspection, we also have that  $\varepsilon_i(\phi_{ij}^{(3)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j)) = -x, -x+A, -x+A+C$  respectively, which implies that  $\phi_{ij}^{(3)}$  preserves  $\varepsilon_i$ . The case of  $\varepsilon_j$  can be done similarly. Then we also have the cases  $\varphi_i$  and  $\varphi_j$  by the formula (2.6). Next, let us show (2) for  $\phi_{ij}^{(3)}$ . We consider the commutativity of  $\phi_{ij}^{(3)}$  and  $\tilde{f}_j$  here. Suppose that  $\tilde{f}_j((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j) = (x)_i \otimes (y-1)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j$ . This is equivalent to the condition  $B < 0$  and  $B+D < 0$ . We consider the cases [I]  $A \leq 0$ , and [II]  $A > 0$ . Set  $(X')_j \otimes (Y')_i \otimes (Z')_j \otimes (U')_i \otimes (V')_j \otimes (W')_i := \phi_{ij}^{(3)} \circ \tilde{f}_j((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j)$ . In the case [I], we can show easily that  $(X')_j \otimes (Y')_i \otimes (Z')_j \otimes (U')_i \otimes (V')_j \otimes (W')_i = (X)_j \otimes (Y)_i \otimes (Z-1)_j \otimes (U)_i \otimes (V)_j \otimes (W)_i$  where  $(X, Y, Z, U, V, W)$  is as above. By using  $A, B \leq 0$ , we can also obtain  $\tilde{f}_j \circ \phi_{ij}^{(3)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j) = (X)_j \otimes (Y)_i \otimes (Z-1)_j \otimes (U)_i \otimes (V)_j \otimes (W)_i$ , which implies  $\tilde{f}_j \circ \phi_{ij}^{(3)} = \tilde{f}_j \circ \phi_{ij}^{(3)}$  in this case. In the case [II], furthermore, we consider the following three cases, (i)  $0 \geq 2B+c_2A \geq 3B+c_2A$ , (ii)  $2B+c_2A \geq 0 \geq 3B+c_2A$ , (iii)  $2B+c_2A \geq 3B+c_2A > 0$ . In each case, we can see the commutativity of  $\phi_{ij}^{(3)}$  and  $\tilde{f}_j$  by inspection. Other cases are shown by similarly. Now we have shown (2), and then it turns out to be that  $\phi_{ij}^{(3)}$  is a strict morphism of crystals.  $\square$

## 4.2 Applications

In this subsection, we introduce an application of the braid-type isomorphisms.

Let  $\mathfrak{g}$  be a semi-simple Lie algebra and  $W$  be the corresponding Weyl group. Here we denote the longest element of  $W$  by  $w_0$ . Let  $s_i$  ( $i \in I$ ) be the simple reflection in  $W$  and  $N$  be the length of the longest element  $w_0$  in  $W$ . Here a sequence  $i_N, i_{N-1}, \dots, i_2, i_1$  is called a *reduced longest word* if  $s_{i_N} s_{i_{N-1}} \cdots s_{i_2} s_{i_1} \in W$  is one of the reduced expressions of  $w_0$ . Here we obtain



**Proposition 4.2** *Let  $\iota = i_N, i_{N-1}, \dots, i_2, i_1$  ( $i_j \in I$ ) be one of the reduced longest words. Then we have*

$$\Psi_\iota(B(\infty)) \subset u_\infty \otimes B_{i_N} \otimes \dots \otimes B_{i_2} \otimes B_{i_1} \cong \mathbf{Z}^N, \quad (4.21)$$

and then for  $\lambda \in P_+$  we also have

$$\Psi_\iota^{(\lambda)}(B(\lambda)) \subset u_\infty \otimes B_{i_N} \otimes \dots \otimes B_{i_2} \otimes B_{i_1} \otimes R_\lambda \cong \mathbf{Z}^N. \quad (4.22)$$

*Remark.* The above proposition implies that the crystals  $B(\infty)$  and  $B(\lambda)$  can be embedded in the  $\mathbf{Z}$ -lattice of the 'finite' rank which is equal to the length of the longest element.

*Proof.* In order to show the proposition, we prepare several lemmas.

**Lemma 4.3** *Let  $\iota = \dots, i_k, \dots, i_2, i_1$  be an infinite sequence of elements in  $I$  such that  $i_l = i_{l-1}$  for some  $l > 1$ . For  $b \in B(\infty)$  set  $\Psi_\iota(b) = (\dots x_k, \dots, x_2, x_1)$ . Then we have  $x_l = 0$  for any  $b \in B(\infty)$ .*

*Proof.* Let us recall the crystal structure of  $\mathbf{Z}_\iota^\infty$  in [16, 2.4]. The action of  $\tilde{f}_i$  is determined by the value  $\sigma_k(b) = x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_{i_j}$  (see (3.10), [16, (2.28)]). Then we have

$$\sigma_{l-1}(b) - \sigma_l(b) = x_{l-1} + x_l \geq 0.$$

By (2.29) in [16], we know that any  $\tilde{f}_i$  never acts on the  $l$ -th component. Thus, we have  $x_l = 0$  since  $\Psi_\iota(b) = \tilde{f}_{i_l} \dots \tilde{f}_{i_1}(\dots, 0, 0)$  if  $b = \tilde{f}_{i_l} \dots \tilde{f}_{i_1} u_\infty$ .  $\square$

**Lemma 4.4** *Let  $\iota = \dots i_k \dots i_2 i_1$  be the infinite sequence such that the first  $N = (\text{length}(w_0))$  subsequence  $i_N, \dots, i_2, i_1$  coincides with one of the reduced longest words. For  $b \in B(\infty)$  set  $(\dots x_k \dots x_2 x_1) := \Psi_\iota(b)$ . Then we have that the  $N + 1$ -th component  $x_{N+1} = 0$  for any  $b \in B(\infty)$ .*

*Proof.* Set  $i = i_{N+1}$  ( $i \in I$ ). By the well-known fact (see e.g., [1]), there exists a reduced longest word  $j_N, \dots, j_1$  such that  $j_N = i$ . By applying the braid-type isomorphisms on the components  $B_{i_N} \otimes \dots \otimes B_{i_2} \otimes B_{i_1}$  properly, we can obtain the new tensor products of crystals  $B_{j_N} \otimes \dots \otimes B_{j_2} \otimes B_{j_1}$  satisfying  $j_N = i$ . Thus, for the new sequence  $\iota' = \dots i_{N+1}, j_N, \dots, j_2, j_1$  setting  $(\dots, x'_k, \dots, x'_2, x'_1) := \Psi_{\iota'}(b)$  we have  $x'_{N+1} = x_{N+1}$  since the braid-type isomorphisms never acts on the  $N + 1$ -th components. Then by Lemma 4.3, we have  $x_{N+1} = x'_{N+1} = 0$ .  $\square$

*Proof of Proposition 4.2.* Let  $\iota$  and  $\vec{x} = (\dots, x_k, \dots, x_2, x_1)$  be as in Lemma 4.4. Let  $m$ -th component in  $\vec{x}$  be the first non-trivial one after  $N$ -th component, that is  $x_m > 0$  and  $x_{N+1} = \dots x_{m-1} = 0$ . By Lemma 4.4,  $m > N + 1$ . Let  $*$  be the map as in 3.1. The element  $b^*$  can be written uniquely:

$$b^* = \dots \tilde{f}_{i_k}^{x_k} \dots \tilde{f}_2^{x_2} \tilde{f}_1^{x_1} u_\infty, \quad (4.23)$$

where  $\tilde{e}_{i_k} \tilde{f}_{i_{k-1}}^{x_{k-1}} \dots \tilde{f}_2^{x_2} \tilde{f}_1^{x_1} u_\infty = 0$  (see [7]). Here note that  $x_{N+1} = \dots = x_{m-1} = 0$ . Adopting the new sequence  $\iota^! := \dots i_m i_N i_{N-1} \dots i_2 i_1$  we have that the  $N + 1$ -th component of  $\Psi_{\iota^!}(b)$  is  $x_m$ . By Lemma 4.4, we have  $x_m = 0$ , which contradicts the definition of  $m$ . Thus, we have  $x_m = 0$ , which implies that all components after  $N$  are just zero. For any sequence  $\dots i_k \dots i_{N+2} i_{N+1}$  the element  $\dots (0)_{i_k} \otimes \dots (0)_{i_{N+2}} \otimes (0)_{i_{N+1}}$  can be identified with  $u_\infty$ . Now, we have completed the proof of the Proposition 4.2.  $\square$

### 4.3 Discussions

In Exampe 3.9, we introduced some counter-example for the positivity assumption of the polyhedral realization. But on the other hand, for the sequence  $\iota_1 = 121321$ , we obtain the image of  $\Psi_{\iota_1}$  by the polyhedral realization. Here applying the braid-type isomorphism to the case for  $\iota_0$  we have the one for  $\iota_1$ , that is, the image of  $\Psi_{\iota_0}$  is given by  $\text{Im}\Psi_{\iota_0} = (\phi_{12}^{(1)})_{456}\text{Im}\Psi_{\iota_1}$ , where the suffix 456 means that  $\phi_{12}^{(1)}$  acts on the 4-th, 5-th and 6-th components of  $\text{Im}\Psi_{\iota_1}$ . Indeed, we have

$$\text{Im}\Psi_{\iota_1} = \{(x_6, \dots, x_2, x_1) \in \mathbf{Z}^6 \mid x_1 \geq 0, x_2 \geq x_4 \geq 0, x_3 \geq x_5 \geq x_6 \geq 0\}.$$

Then using the explicit form of  $\phi_{12}^{(1)}$  as in Proposition 4.1, we get

$$\text{Im}\Psi_{\iota_0} = \{(x_6, \dots, x_2, x_1) \in \mathbf{Z}^6 \mid \begin{array}{l} x_1 \geq 0, x_4 \geq 0, x_3 \geq x_4 + x_6, \\ x_2 + x_4 \geq x_5 \geq x_6 \geq 0, x_2 \geq x_6. \end{array} \}.$$

For dominant integral weight  $\lambda = m_1\Lambda_1 + m_2\Lambda_2 + m_3\Lambda_3$  we also have by Theorem 3.8,

$$\text{Im}\Psi_{\iota_1}^{(\lambda)} = \{(x_6, \dots, x_2, x_1) \in \mathbf{Z}^6 \mid \begin{array}{l} m_1 \geq x_1 \geq 0, x_2 \geq x_4 \geq 0, x_3 \geq x_5 \geq x_6 \geq 0, \\ m_2 \geq x_2 - x_1, x_4, m_3 \geq x_3 - x_2, x_5 - x_4, x_6. \end{array} \}.$$

Moreover, by applying  $(\phi_{12}^{(1)})_{456}$  to this we have

$$\text{Im}\Psi_{\iota_0}^{(\lambda)} = \{(x_6, \dots, x_2, x_1) \in \mathbf{Z}^6 \mid \begin{array}{l} m_1 \geq x_1 \geq 0, m_3 \geq x_4 \geq 0, x_3 \geq x_4 + x_6, \\ x_2 + x_4 \geq x_5 \geq x_6 \geq 0, x_2 \geq x_6, \\ m_2 \geq x_2 - x_1, x_5 - x_4, x_6, m_3 \geq x_3 - x_2. \end{array} \}.$$

Now we know that the images of  $\Psi_{\iota_0}$  and  $\Psi_{\iota_0}^{(\lambda)}$  is realized in the polyhedral convex cone or convex polytope in  $\mathbf{Z}^6$ . But in general, it is not clear since the braid-type isomorphisms are not linear but piece-wise linear. (For A-type it seems to be correct.). Our further problem is to describe explicitly the image of  $\Psi_{\iota}$  and  $\Psi_{\iota}^{(\lambda)}$  for an arbitrary reduced longest word  $\iota$ .

## References

- [1] Bourbaki N, Groupes et algèbres de Lie, Masson, Paris, 1981.
- [2] Berenstein A and Zelevinsky A, Canonical bases for the quantum group of type  $A_r$  and piecewise-linear combinatorics, *Duke Math J.*, **82** (1996), 473-502.
- [3] Jimbo M, Misra K.C, Miwa T and Okado M, Combinatorics of representations of  $U_q(\widehat{\mathfrak{sl}(n)})$  at  $q = 0$ , *Comm. Math. Phys.*, **136** (1991), 543-566.
- [4] Joseph A, Quantum groups and their primitive ideals, Springer-Verlag, (1995).
- [5] Kashiwara M, Crystallizing the  $q$ -analogue of universal enveloping algebras, *Comm. Math. Phys.*, **133** (1990), 249-260.

- [6] Kashiwara M, On crystal bases of the  $q$ -analogue of universal enveloping algebras, *Duke Math. J.*, **63** (1991) 465–516.
- [7] Kashiwara M, Crystal base and Littelmann’s refined Demazure character formula, *Duke Math. J.*, **71**(1993), 839–858.
- [8] Kashiwara M, Crystallization of quantized enveloping algebras, *Sugaku Expositions*, **7**, No.1, June, 1994.
- [9] Kang S-J, Kashiwara M, Misra K, Miwa T, Nakashima T and Nakayashiki A, Affine crystals and vertex models, *Int.J.Mod.Phys.*, A7 Suppl. 1A (1992) 449–484.
- [10] Kang S-J, Kashiwara M, Misra K, Miwa T, Nakashima T and Nakayashiki A, Perfect crystals of quantum affine Lie algebras, *Duke Math. J.*, **68** (1992), 499–607.
- [11] Kashiwara M and Nakashima T, Crystal graph for representations of the  $q$ -analogue of classical Lie algebras, *J. Algebra*, **165**, (1994), 295–345.
- [12] Littelmann P, A Littlewood-Richardson type rule for symmetrizable Kac-Moody algebras, *Invent. Math.*, **116** (1994), 329–346.
- [13] Littelmann P, Path and root operators in representation theory, *Ann. of Math.*(2) **142** (1995), no. 3, 499–525.
- [14] Littelmann P, Cones, Crystals, and Patterns, *Transformation Groups*, Vol.3, No.2, 1998, 145–179.
- [15] Nakashima T, Crystal Base and a Generalization of the Littlewood-Richardson Rule for the Classical Lie Algebras, *Commun. Math. Phys.*, **154**, (1993), 215–243.
- [16] Nakashima T and Zelevinsky A, Polyhedral Realization of Crystal Bases for Quantized kac-Moody Algebras, *Advances in Mathematics*, **131**, No.1, (1997), 253–278.
- [17] Nakashima T., Polyhedral Realizations of Crystal Bases for Integrable Highest Weight Modules, preprint, mathQA/9806085 (to appear in *J.Algebra*).